United Kingdom
Mathematics Trust

Intermediate Mathematical Olympiad

Cayley paper

Monday 15 March 2021

supported by [XTX] MARKETS Overleaf

England & Wales: Year 9 or below
Scotland: S2 or below
Northern Ireland: Year 10 or below

These problems are meant to be challenging! The earlier questions tend to be easier; later questions tend to be more demanding.

Do not hurry, but spend time working carefully on one question before attempting another.

Try to finish whole questions even if you cannot do many: you will have done well if you hand in full solutions to two or more questions.

You may wish to work in rough first, then set out your final solution with clear explanations and proofs.

Instructions

1. Do not open the paper until the invigilator tells you to do so.
2. Time allowed: **2 hours**.
3. The use of blank or lined paper for rough working, rulers and compasses is allowed; **squared paper, calculators and protractors are forbidden**.
4. Write on one side of the paper only and start each question on a fresh sheet.
5. Write your participant ID and question number neatly in the top left corner of each page and arrange them with your cover sheet on top, so that your teacher can easily upload them to the marking platform. **Do not hand in rough work**.
6. Your answers should be fully simplified, and exact. They may contain symbols such as π, fractions, or square roots, if appropriate, but not decimal approximations.
7. You should give full written solutions, including mathematical reasons as to why your method is correct. Just stating an answer, even a correct one, will earn you very few marks; also, incomplete or poorly presented solutions will not receive full marks.

Enquiries about the Intermediate Mathematical Olympiad should be sent to:

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- ◇ *Try to finish whole questions even if you cannot do many.*
- ◇ *You will have done well if you hand in full solutions to two or more questions.*
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- ◇ *Give full written solutions, including mathematical reasons as to why your method is correct.*
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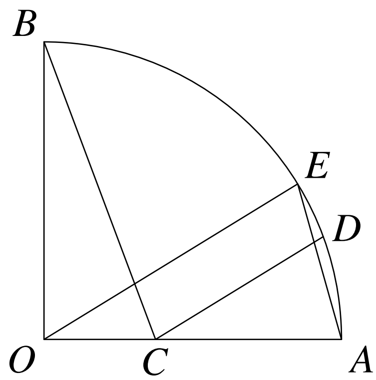
- ◇ *Incomplete or poorly presented solutions will not receive full marks.*
- ◇ *Do* not *hand in rough work.*

1. In the four-digit number 4753, three two-digit numbers are formed by successive pairs of digits $(47, 75, 53)$. Exactly two of these two-digit numbers are prime. Find all four-digit numbers in which all four digits are prime, and **all three** two-digit numbers formed by successive digits are prime.

2. Jack has a large number of tiles, each of which is in the shape of a right-angled triangle with side lengths 3 cm, 4 cm and 5 cm. Is it possible for Jack to combine a number of these tiles, without gaps or overlap, to form a rectangle of size 2016 cm by 2021 cm?

3. In the diagram, OAB is a quarter circle, OE and CD are parallel, BC and EA are parallel, and $\angle BCD = 4 \times \angle OBC$.

What is the size of $\angle OBC$?



4. Let $S(n)$ denote the sum of the first n terms of the series

$$1 - 2 + 3 - 4 + 5 - 6 + \cdots \quad .$$

For example, $S(5) = 1 - 2 + 3 - 4 + 5 = 3$.

For what values of a and b does $S(a) + S(b) + S(a + b) = 1$?

5. Real numbers p, q, r, x, y, z satisfy the equations

$$\frac{x}{p} + \frac{q}{y} = 1,$$

$$\frac{y}{q} + \frac{r}{z} = 1.$$

Prove that $pqr + xyz = 0$.

6. During an early morning drive to work, Simon encountered n sets of traffic lights, each set being red, amber or green as he approached it. He noticed that consecutive lights were never the same colour.

Given that he saw *at least* two red lights, find a simplified expression, in terms of n, for the number of possible sequences of colours Simon could have seen.